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EVOLUTION OF THE INTERFACE IN A STRATIFIED ANISOTROPIC POROUS MATERIAL

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It is shown that the boundary-value problem describing the evolution of the interface during impregnation of a stratified inhomogeneous anisotropic porous material with a viscous fluid can be reduced to a similar problem for a stratified inhomogeneous isotropic material by nonorthogonal transformation of the coordinates. As a result, the well-known estimates of the problem parameters determining the interface configuration for impregnation of an isotropic material can be extended to the anisotropic case.

Key words: *free-boundary problem, multiphase media, bulk anisotropy.*

Introduction. During manufacture of composite materials using the vacuum-assisted resin transfer molding technology, it is necessary to solve the problem of filtration penetration of a viscous fluid into a porous material originally filled with air under the assumption that each phase is cohesive and the phases are separated by a distinct interface [1]. The significant factors determining filtration are the stratified inhomogeneity of the porous material and the bulk anisotropy of the layers. For the case of a stratified isotropic material, asymptotic estimates of the problem parameters determining the interface configuration in the steady-state mode were obtained in [2]. In [3], it was established that the problem of penetration of a viscous fluid into a homogeneous anisotropic porous material can be reduced to the same problem for an isotropic material by means of the well-known nonorthogonal coordinate transformation. In the present paper, we show the effectiveness of such coordinate transformation for the case of a stratified anisotropic porous material.

The impregnation process of a porous material is shown schematically in Fig. 1. It is assumed that the filtration process does not depend on the coordinate whose unit vector is perpendicular to the plane (x, y), and, hence, the problem can be considered two-dimensional. The material consists of two layers (upper and lower). The parameters of the upper layer will be denoted by subscript plus, and the parameters of the lower layer by subscript minus. The layers have thickness $h^{\pm}$ and porosity $m^{\pm}$. Each layer is homogeneous and anisotropic, and the main axes of the permeability tensor coincide with the Cartesian axes x, y (the choice of the coordinate origin is arbitrary). Then, the permeability tensor in each layer is completely characterized by the diagonal components $k^{\pm}_{xx}$ and $k^{\pm}_{yy}$. The degree of anisotropy of the layers is characterized by the dimensionless parameters $\chi^{\pm}$ [4]:

$$\chi^{\pm} = \sqrt{k^{\pm}_{yy}/k^{\pm}_{xx}}.$$

According to the technology considered, the bottom of the lower layer and the top of the upper layer are impermeable [1]. The left boundary of the material is connected to a tank with a viscous fluid maintained at constant pressure. The right boundary of the material is connected to an air chamber. As the chamber pressure decreases, a pressure difference arises in the fluid, resulting in filtration impregnation of the material.

There are two stages of the impregnation process: an early stage in which the size of the impregnation zone in the horizontal direction is comparable to the total thickness of the material $(h^+ + h^-)$, and a late stage in which the size of this zone is significantly larger than the value of $h^+ + h^-$ and, hence, the length of the material in the

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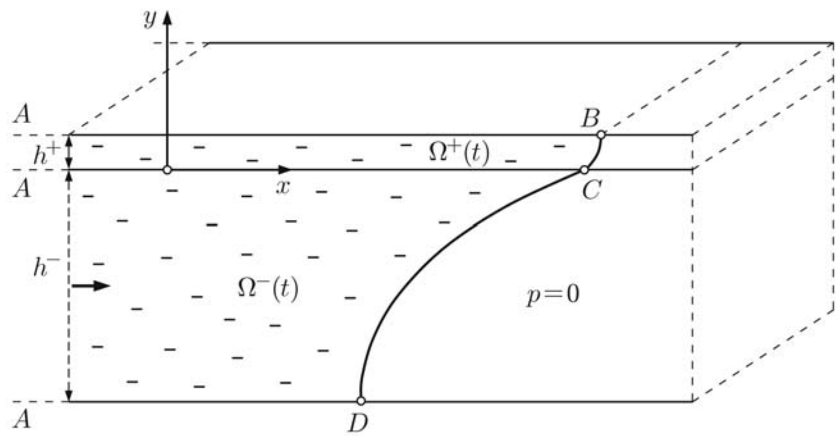

Fig. 1. Diagram of impregnation of a two-layer porous material with a viscous fluid.

left horizontal direction can be considered infinite. In the present paper, we consider in detail only the late, more informative, stage of impregnation. In particular, in this stage, the interface shape is established. The early stage of the impregnation process is briefly discussed at the end of Sec. 2.

1. Formulation of the Problem in Terms of the Pressure Function. In the region of the porous material occupied by air, the pressure is constant. The region of the material occupied by the fluid will be denoted by $\Omega^\pm(t)$ (see Fig. 1). It is assumed that the fluid is incompressible and its motion in the region $\Omega^\pm(t)$ is described by the Darcy law [1, 4]:

$$\Omega^\pm(t): \qquad \frac{\partial v_x^\pm}{\partial x} + \frac{\partial v_y^\pm}{\partial y} = 0, \qquad (v_x^\pm, v_y^\pm) = -\mu^{-1}\Big(k_{xx}^\pm \frac{\partial p^\pm}{\partial x},\ k_{yy}^\pm \frac{\partial p^\pm}{\partial y}\Big). \tag{1.1}$$

Here μ is the fluid viscosity and $\boldsymbol{v}^\pm = (v_x^\pm, v_y^\pm)$ is the filtration velocity. Then, the pressure distribution $p^\pm(x,y,t)$ in each layer will satisfy the equations

$$\Omega^\pm(t): \qquad k_{xx}^\pm \frac{\partial^2 p^\pm}{\partial x^2} + k_{yy}^\pm \frac{\partial^2 p^\pm}{\partial y^2} = 0. \tag{1.2}$$

As noted above, the boundaries AB and AD are impermeable:

$$AB \cup AD: \qquad \frac{\partial p^\pm}{\partial y} = 0. \tag{1.3}$$

At the interface between the layers AC, the continuity conditions for the pressure and the normal component of the filtration velocity are satisfied [1]:

$$AC: \qquad p^+ = p^-, \qquad k_{yy}^+ \frac{\partial p^+}{\partial y} = k_{yy}^- \frac{\partial p^-}{\partial y}. \tag{1.4}$$

The interface $BC \cup CD$ is free and, therefore, two boundary conditions should be specified on this interface. The first, dynamic, condition assumes that the fluid pressure on the interface is equal to the capillary pressure P_c, which is constant in the case of a homogeneous porous medium [4, 5]. At the same time, this is the only condition of the problem that contains the absolute value of the pressure. By the quantity $p^\pm(x,y,t)$ is meant the relative pressure, i.e., the pressure reckoned from the value of P_c. Then, the dynamic condition can be written as [1, 2]

$$BC \cup CD: \qquad p^\pm = 0. \tag{1.5}$$

The second, kinematic, condition is a consequence of the materiality of the free boundary and, in view of the dynamic condition (1.5), it can be written as

$$BC \cup CD: \qquad \frac{dp^\pm}{dt} = 0,$$

where d/dt is the substantial time derivative. It should be noted that the last condition is not identical to condition (1.5) (this condition is discussed in [3, 6] for the Hele-Shaw problem, and in [7] for the related Stefan problem).

We represent the substantial time derivative according to the well-known formula [8], taking into account that the velocity of material particles of the fluid belonging to the region $BC \cup CD$ is equal to $\boldsymbol{v}^\pm/m^\pm$ [4]. As a result, using formula (1.1) for the filtration velocity component, we obtain the kinematic condition

$$BC \cup CD: \qquad \frac{\mu m^\pm}{k_{xx}^\pm}\frac{\partial p^\pm}{\partial t} = \Big(\frac{\partial p^\pm}{\partial x}\Big)^2 + (\chi^\pm)^2\Big(\frac{\partial p^\pm}{\partial y}\Big)^2. \tag{1.6}$$

To close the problem, it is necessary to write the initial condition and to specify the condition at infinity on the left. The process considered corresponds to the case of fluid flow with an evolving free boundary; the flow is creeping, and time derivatives in the governing equation (1.2) are absent. Therefore, by analogy with the Hele-Shaw problem, the initial condition can be taken to be the configuration of the region at the initial time:

$$t = 0: \qquad \Omega^\pm(t) = \Omega_0^\pm. \tag{1.7}$$

By virtue of conditions (1.3) and (1.4), with distance from the free surface, the pressure across both layers is equalized [4, 5]. Accordingly, the condition at infinity can be specified in the form of a stationary pressure gradient

$$x \to -\infty: \qquad \frac{\partial p^\pm}{\partial x} = -\lambda^2. \tag{1.8}$$

Thus, the evolution of the interface during impregnation of a two-layer anisotropic porous material is described by the boundary-value problem (1.2)–(1.8). The special case $\chi^\pm = 1$, i.e., $k_{xx}^\pm = k_{yy}^\pm = k^\pm$, corresponds to a two-layer isotropic material. Mathematically, problem (1.2)–(1.8) is a generalization of the unilateral idealized Hele-Shaw problem [9, 10] that takes into account the stratified inhomogeneity and bulk anisotropy of the porous medium.

2. Nonorthogonal Coordinate Transformation. We consider the transformation of the coordinates x, y to the new coordinates X, Y [3]:

$$X = x, \qquad Y = y/\chi^\pm, \tag{2.1}$$

and the transformation of the functions $p^\pm(x, y, t)$ to the functions $P^\pm(X, Y, t)$:

$$P^\pm(X, Y, t) = p^\pm(x, y, t)\Big|_{\substack{x = x(X)\\ y = y(Y)}}. \tag{2.2}$$

From formula (2.1), it follows that, in the vertical direction, the upper and lower layers are stretched differently. For the regions $\Omega^\pm(t)$ in the space X, Y, we retain the former notation.

As a matter of fact, (2.1) is the well-known nonorthogonal coordinate transformation [4, 11, 12] which reduces Eqs. (1.2) to the Laplace equations

$$\Omega^\pm(t): \qquad \Delta P^\pm = 0. \tag{2.3}$$

Here the Laplace operator is written in the coordinates X and Y. Equations (2.3) allow the penetration of a viscous fluid into a two-layer anisotropic porous material to be treated as the penetration of the same fluid into a two-layer isotropic porous material in the space X, Y, where and $P^\pm(X, Y, t)$ have the meaning of the pressure distribution function in this space. The thicknesses of the layers are directly defined by formula (2.1):

$$H^\pm = h^\pm/\chi^\pm. \tag{2.4}$$

In the space X, Y, the parameters of the layers of the two-layer isotropic porous material, namely, the permeability $K^\pm$ and porosity $M^\pm$ layers, can be defined as follows:

$$K^\pm = k_{xx}^\pm \chi^\pm, \qquad M^\pm = m^\pm \chi^\pm. \tag{2.5}$$

Applying transformation (2.1), (2.2), (2.4), (2.5) to the boundary and initial conditions of the boundary-value problem (1.2)–(1.8), we obtain the boundary and initial conditions for the functions $P^\pm(X, Y, t)$:

$$\begin{aligned} &AB \cup AD: && \frac{\partial P^\pm}{\partial Y} = 0,\\ &AC: && P^+ = P^-, \qquad K^+\frac{\partial P^+}{\partial Y} = K^-\frac{\partial P^-}{\partial Y},\\ &BC \cup CD: && P^\pm = 0; \end{aligned} \tag{2.6}$$

$$BC \cup CD: \qquad \frac{\mu M^{\pm}}{K^{\pm}} \frac{\partial P^{\pm}}{\partial t} = \Big(\frac{\partial P^{\pm}}{\partial X}\Big)^2 + \Big(\frac{\partial P^{\pm}}{\partial Y}\Big)^2,$$
$$t = 0: \qquad \Omega^{\pm}(t) = \Omega_0^{\pm}; \tag{2.7}$$

$$X \to -\infty: \qquad \frac{\partial P^{\pm}}{\partial X} = -\lambda^2. \tag{2.8}$$

A comparison of the boundary-value problem (2.3), (2.6)–(2.8) with problem (1.2)–(1.8) shows that transformations (2.1), (2.2), (2.4), and (2.5) allow the boundary-value problem (1.2)–(1.8) of the evolution of the interface during impregnation of a two-layer anisotropic porous material to be reduced to the similar problem for a two-layer isotropic porous material. This result is also valid for the early stage of impregnation. In this case, the porous material at the left has a rectilinear boundary $x = 0$ (to which we attach the coordinate origin). According to the process technology described, the fluid pressure on the interface is constant:

$$x = 0: \qquad p^{\pm} = p_0 > 0. \tag{2.9}$$

Thus, in the boundary-value problem (1.2)–(1.8), condition (1.8) at infinity needs to be replaced by condition (2.9) on the boundary $x = 0$. Invariance of the latter with respect to transformation (2.1), (2.2), (2.4), (2.5) is obvious. Consequently, the conclusion that the problem of impregnation of a stratified anisotropic material is reducible to the isotropic case is also valid for the early stage of impregnation.

3. Case of Steady-State Boundary. Let the shape of the free boundary be steady-state, i.e., the process enters the traveling-wave mode and the contour $DC \cup CB$ moves along the x axis with constant velocity u without shape changes. Then, the pressure functions $p^{\pm}(x, y, t)$ have the structure $p^{\pm}(x - ut, y)$, and, hence,

$$\frac{\partial p^{\pm}}{\partial t} = -u \frac{\partial p^{\pm}}{\partial x}.$$

Accordingly, from (1.6) it is possible to obtain the material condition for the free boundary in the steady-state mode

$$BC \cup CD: \qquad \frac{\mu m^{\pm} u}{k_{xx}^{\pm}} \frac{\partial p^{\pm}}{\partial x} + \Big(\frac{\partial p^{\pm}}{\partial x}\Big)^2 + (\chi^{\pm})^2 \Big(\frac{\partial p^{\pm}}{\partial y}\Big)^2 = 0. \tag{3.1}$$

In this formulation, the velocity of motion of the interface u is a nonfree parameter of the problem: it is linked to the other process parameters by balance relations. We prove this statement by calculating the fluid flux through the cross section at infinity q using the Darcy law (1.1) and the known pressure gradient (1.8):

$$q = \lambda^2 (k_{xx}^{+} h^{+} + k_{xx}^{-} h^{-})/\mu.$$

At the same time, because the interface configuration (traveling wave) is invariant, we can write $q = u(m^{+}h^{+} + m^{-}h^{-})$. Equating these two expressions for q, we obtain the velocity of motion of the contour u expressed in terms of the determining process parameters:

$$u = \frac{\lambda^2}{\mu} \frac{k_{xx}^{+} h^{+} + k_{xx}^{-} h^{-}}{m^{+}h^{+} + m^{-}h^{-}}. \tag{3.2}$$

Thus, the motion of the interface during steady-state fluid penetration into a two-layer anisotropic porous material is described by the boundary-value problem (1.2)–(1.5), (1.8), (3.1), in which the parameter u is given by formula (3.2). The variant $\chi^{\pm} = 1$, i.e., $k_{xx}^{\pm} = k_{yy}^{\pm} = k^{\pm}$, which corresponds to the case of a two-layer isotropic porous material, is considered in [2].

As in [2], we introduce two dimensionless parameters δ and ε, which characterize the relations between the thickness of the layers and their permeabilities:

$$\delta = \frac{m^{+}h^{+}}{m^{-}h^{-}}, \qquad \varepsilon = \frac{k_{xx}^{-} h^{-}}{k_{xx}^{+} h^{+}}. \tag{3.3}$$

In the relations of the permeabilities, only the longitudinal permeabilities $k_{xx}^{\pm}$ of the layers are assumed to be significant because the transverse permeabilities $k_{yy}^{\pm}$ do not influence the velocity of motion of the contour with a steady-state shape [see formula (3.2)].

If the characteristic velocity is taken to be the fluid particle velocity at infinity in the lower layer

$$u_{\infty} = \frac{\lambda^2 k_{xx}^{-}}{\mu m^{-}}, \tag{3.4}$$

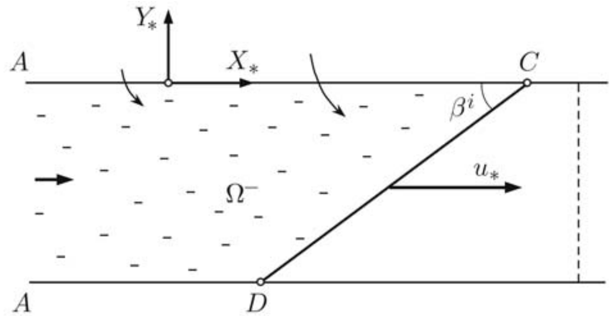

Fig. 2. Approximate configuration of the interface during steady-state impregnation of a two-layer isotropic porous material in the case of large differences in thicknesses and permeability between the layers.

then, it is possible to introduce the dimensionless velocity of motion of the interface $u_* = u/u_\infty$. According to formula (3.2), the velocity u_* is completely determined by the quantities δ and ε, as in [2]:

$$u_* = \frac{1+\varepsilon^{-1}}{1+\delta}. \tag{3.5}$$

We apply transformation (2.1), (2.2), (2.4), (2.5) to the boundary-value problem (1.2)–(1.5), (1.8), (3.1). According to Sec. 2, formulas (1.2)–(1.5) and (1.8) are transformed to (2.3), (2.6), and (2.8). The material condition for the free boundary (3.1) becomes

$$BC \cup CD: \qquad \frac{\mu M^{\pm} u}{K^{\pm}} \frac{\partial P^{\pm}}{\partial X} + \Big(\frac{\partial P^{\pm}}{\partial X}\Big)^2 + \Big(\frac{\partial P^{\pm}}{\partial Y}\Big)^2 = 0. \tag{3.6}$$

Thus, the specified transformation allows the boundary-value problem (1.2)–(1.5), (1.8), (3.1) to be reduced to the similar problem (2.3), (2.6), (2.8), (3.6) for the case of a two-layer isotropic porous material.

Let us verify that the transformation does not change the physical meaning of parameters (3.2)–(3.4). Formula (3.2) is transformed to

$$u = \frac{\lambda^2}{\mu} \frac{K^+H^+ + K^-H^-}{M^+H^+ + M^-H^-}. \tag{3.7}$$

A comparison of formulas (3.1) and (3.2) with formulas (3.6) and (3.7) shows that the physical meaning of the parameter u (the velocity of motion of the interface in the space X, Y) remains unchanged.

Similarly, the characteristic velocity u_∞ retains the physical meaning of the velocity of material particles of the fluid at infinity in the lower layer of the material in the space X, Y:

$$u_\infty = \frac{\lambda^2 K^-}{\mu M^-}.$$

As regards the dimensionless parameters δ and ε, they are invariant with respect to the specified transformation:

$$\delta = \frac{m^+h^+}{m^-h^-} = \frac{M^+H^+}{M^-H^-}, \qquad \varepsilon = \frac{k^-_{xx}h^-}{k^+_{xx}h^+} = \frac{K^-H^-}{K^+H^+}.$$

As a consequence, the dimensionless velocity of motion of the boundary u_* [see formula (3.5)] is also invariant with respect to transformation (2.1), (2.2), (2.4), (2.5).

The asymptotic analysis in [2] of the motion of the interface during steady-state impregnation of a two-layer isotropic porous material corresponds to the case of a large difference in thickness between the layers (which is characterized by the smallness of the parameter $\delta \ll 1$) and their permeabilities (which is characterized by the smallness of the parameter $\varepsilon \ll 1$), which is of greatest interest from the viewpoint of technological applications [1]. Obviously, by means of the transformation which is inverse to (2.1), (2.2), (2.4), (2.5), the results of the asymptotic analysis [2] can be extended to the impregnation of a two-layer anisotropic porous material for $\delta \ll 1$ and $\varepsilon \ll 1$.

4. Extension of the Results of [2] to the Impregnation of a Two-Layer Anisotropic Porous Material. We briefly consider the results of [2]. In the case $\delta \ll 1$ and $\varepsilon \ll 1$, the motion of the interface during steady-state impregnation of a two-layer isotropic porous material is described by a separate boundary-value problem for the lower layer (Fig. 2):

$$\Omega^-: \qquad \frac{\partial^2 P_*}{\partial X_*^2} + \frac{\partial^2 P_*}{\partial Y_*^2} = 0,$$

$$AD: \qquad \frac{\partial P_*}{\partial Y_*} = 0, \tag{4.1}$$

$$CD: \qquad P_* = 0, \qquad u_* \frac{\partial P_*}{\partial X_*} + \Big(\frac{\partial P_*}{\partial X_*}\Big)^2 + \Big(\frac{\partial P_*}{\partial Y_*}\Big)^2 = 0;$$

$$AC: \qquad \frac{\partial^2 P_*}{\partial X_*^2} + \varepsilon \frac{\partial P_*}{\partial Y_*} = 0, \qquad \Big(\frac{\partial P_*}{\partial X_*}, \frac{\partial P_*}{\partial Y_*}\Big)\Big|_A = (-1, 0). \tag{4.2}$$

Here X_*, Y_*, and P_* are dimensionless variables:

$$X_* = \frac{X}{H^-}, \qquad Y_* = \frac{Y}{H^-}, \qquad P_* = \frac{P^-}{\lambda^2 H^-}.$$

As the characteristic length we use the thickness of the layer H^-, and as the characteristic pressure the quantity $\lambda^2 H^-$ [see (2.8)].

We note that condition (4.2) on the boundary AC characterizes the fluid flow from the highly permeable upper layer. The asymptotic analysis [2] is based on the replacement of this boundary condition with the simpler condition

$$AC: \qquad \frac{\partial P_*}{\partial X_*} = -1, \tag{4.3}$$

which follows from (4.2) for $\varepsilon = 0$. From the solution of the auxiliary boundary-value problem (4.1), (4.3), we obtain the main term of the asymptotic expansion of the solution of the complete boundary-value problem (4.1), (4.2) in powers of $\sqrt{\varepsilon}$. This solution was constructed analytically, which made it possible to analyze the effect of the determining parameters of the problem on the configuration of the free boundary. In particular, it was found that the interface configuration is well approximated by the straight-line segment which intersects the horizontal at the angle β^i (see Fig. 2) which is determined only by the quantity u_* [2]:

$$\beta^i = \arctan\,(1/\sqrt{u_* - 1}\,). \tag{4.4}$$

Taking into account the smallness of the parameters δ and ε and using formula (3.5), it is possible to estimate the value of this angle and the length of the interface along the horizontal:

$$\beta^i = \sqrt{\varepsilon}\,[1 + O(\sqrt{\varepsilon}\,)], \qquad X_C - X_D = \frac{H^-}{\sqrt{\varepsilon}}\,[1 + O(\sqrt{\varepsilon}\,)].$$

Because the transition from the isotropic case to the anisotropic case consists of stretching the layers along the vertical [see formulas (2.1) and (2.4)], for an anisotropic material, a sufficiently accurate approximation of the interface configuration is also the straight-line segment which intersects the horizontal at the angle β^a in the plane (x, y). To find the angle, we take into account that, according to formula (2.4), the dimensional thickness of the lower layer is calculated from the formula

$$h^- = H^- \chi^-.$$

Then, formula (4.4) directly leads to the formula for the angle β^a

$$\beta^a = \arctan\,(\chi^-/\sqrt{u_* - 1}\,).$$

Thus, this angle and the length of the interface along the horizontal can be estimated as

$$\beta^a = \chi^- \sqrt{\varepsilon}\,[1 + O(\sqrt{\varepsilon}\,)], \qquad x_C - x_D = \frac{h^-}{\chi^- \sqrt{\varepsilon}}\,[1 + O(\sqrt{\varepsilon}\,)].$$

Similarly, it is possible to extend the results for the steady-state impregnation of a two-layer isotropic porous material to the anisotropic case for considerable ε. However, such results are currently not available: for $\varepsilon \approx 1$, the problem can be solved only numerically. In addition, the nonstandard boundary condition (4.2) and the presence of a sharp angle at the point C near the boundary of the region considerably complicate the numerical analysis of the free-boundary problem (4.1), (4.2).

Conclusions. It was shown that the results of the mathematical analysis of the impregnation of a stratified isotropic porous material with a viscous fluid can easily be extended to the case of a stratified anisotropic material. Using the results of asymptotic analysis [2], which are primarily of theoretical interest, estimates of practical interest were obtained for the problem parameters determining the steady-state configuration of the interface of a stratified anisotropic material.

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